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A SIMPLE ONE-DIMENSIONAL FINITE ELEMENT ALGORITHM WITH MULTI-DIMENSIONAL CAPABILITIES

by

D. W. Pepper and A. J. Baker*

Savannah River Laboratory
E. I. du Pont de Nemours and Company
Aiken, South Carolina 29801, U.S.A.

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* Present address: Associate Professor, Department of Engineering Science and Mechanics, University of Tennessee, Knoxville, Tennessee 37916.

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DISCLAIMER

A SIMPLE ONE-DIMENSIONAL FINITE ELEMENT ALGORITHM WITH MULTI-DIMENSIONAL CAPABILITIES*

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ABSTRACT — A simple finite element algorithm is derived for the solution of the one-dimensional advection-diffusion transport equation. Local interpolation functionals are obtained for each subdomain, or element, by using linear basis functions. The integral form of the finite element algorithm is evaluated by using hypermatrices and a natural coordinate system. Assemblage over two adjacent elements gives the tridiagonal recursion relation for chapeau functions. By time-splitting either a two- or three-dimensional equation into a series of one-dimensional equations and by using the basic one-dimensional finite element algorithm, multi-dimensional problems can be solved without additional programming complexity or storage.

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† Present address: Associate Professor, Department of Engineering Science and Mechanics, University of Tennessee, Knoxville, TN 37916.

NOMENCLATURE

a_1, a_3	boundary constraint constants
c	transport variable
c'	$\frac{\partial c}{\partial t}$
c*	approximation [~~function~~] of c
$c^{\dagger}$, $c^{\dagger\dagger}$	intermediate values of c
g	amplification factor
j	$\sqrt{-1}$
k	diffusion coefficient
L	wavelength
L(c)	differential operator
$\ell(c)$	boundary constraint
L_m	length of line elements
$\hat{n}$	unit vector normal to the surface
n_1, n_2	natural coordinate exponents
R_m	element set
S_m	assembly algorithm
s	nondimensional space
t	time
U, V, W	velocity components
U*	approximation [~~function~~] of U
x, y, z	direction components

GREEK SYMBOLS

α	$k\Delta t/2\Delta x^2$
Δt	time-step increment
Δx	constant element length
Δx_+	$x_i - x_{i-1}$
Δx_-	$x_{i+1} - x_i$
λ	wavenumber ($2\pi/L$)
μ	time-integration parameter
ξ_1, ξ_2	natural coordinates
ρ	separation constant
σ	Courant number ($U\Delta t/\Delta x$)
τ	nondimensional time
$\phi(x)$, ϕ	basis function (function of space)

SUBSCRIPTS

i-1, i, i+1	nodal point values at x_{i-1}, x_i, x_{i+1}
m	elements variable

SUPERSCRIPTS

n, n+1	time level

SYMBOLS

{ }	column matrix
[ ]	row matrix
[ ]	square matrix

INTRODUCTION

The application of the finite element procedure for the solution of partial differential equations is gaining widespread acceptance. The ability of the finite element procedure to solve problems that are arbitrarily shaped as well as the alleviation of boundary condition problems associated with conventional finite difference procedures is well known. By using local interpolation functionals over each subdomain, or element, a set of linearized algebraic equations are obtained. These equations can be solved by any direct, iterative, or inverse numerical technique. Subsequent use of an explicit or implicit integration procedure permits closure of the solution over the global domain. Unfortunately, the more commonly used technique of using triangular or isoparametric elements in two-dimensional space (tetrahedral or cubic in three dimensions) usually requires excessive coding and in some instances excessive core and computational time. In order to overcome the burden of computer programming, as well as core requirements and computational time, a simple finite element algorithm is derived which has the appearance of a tridiagonal recursion finite difference technique.

The algorithm is derived for the one-dimensional time-dependent advection diffusion equation by using linear basis functions (simplest finite element interpolation scheme). Because the basis functions are formed over two adjacent grid intervals, they have the appearance of peaked hats, or "chapeau" functions (commonly referred by geophysical fluid dynamists). Examples

of the use of chapeau function solutions to geophysical problems are reported by Raymond and Garder [1], Long and Hicks [2]. Pinder and Gray [3], Long and Pepper [4], and Price, Cavendish, and Varga [5]. By time-splitting either the two or three-dimensional equation into a series of one-dimensional equations and using the basic one-dimensional algorithm, multi-dimensional problems can be solved without additional programming complexity or storage. A three-dimensional, time-split chapeau function procedure is discussed by Pepper and Kern [6].

DERIVATION OF THE ALGORITHM

The differential operator, L(c), is defined as

$$L(c) \equiv \frac{\partial c}{\partial t} + U \frac{\partial c}{\partial x} - \frac{\partial}{\partial x}\left(k \frac{\partial c}{\partial x}\right) = 0 \tag{1}$$

with the boundary constraint, $\ell(c)$, given as

$$\ell(c) \equiv a_1 c + k \frac{\partial c}{\partial x} \cdot \hat{n} - a_3 = 0 \tag{2}$$

where c is the unknown variable, U is the velocity, k is the diffusion coefficient, $\hat{n}$ is the unit vector normal to the surface, and a_1 and a_3 are constants. For $a_1 = a_3 = 0$, the flux is computed as zero at the surface.

The lowest degree approximation valid for c(x,t) is a linear function, $c(x,t) = a(t) + b(t)x$. Thus

$$c^*(x,t) = \lfloor \phi(x) \rfloor \{c(t)\}_m \tag{3}$$

where the asterisk denotes the approximation function, and the subscript m denotes belonging to a finite element interval. The

brackets indicate a row matrix, and the braces denote a column matrix. If k is assumed to be constant, the terms in equation (1) are approximated as

$$\frac{\partial c^*(x,t)}{\partial x} = \frac{d}{dx} \lfloor \phi(x) \rfloor \{c(t)\}_m \tag{4}$$

$$\frac{\partial c^*(x,t)}{\partial t} = \lfloor \phi(x) \rfloor \{c(t)\}'_m \tag{5}$$

$$U^*(x) = \lfloor \phi(x) \rfloor \{U\}_m \tag{6}$$

The Galerkin weighted residual method requires that the solution error be made orthogonal to the space of the function set. Integrating over the solution domain, $R_m \cup \partial R_m$, and the domain closure, $R_m \cap \partial R$, yields ($\phi(x) \equiv \phi$, $c(t) \equiv c$ for simplicity)

$$\int_{R_m} \{\phi\}_m L(c^*)_m d\tau - \rho \int_{\partial R_m \cap \partial R} \{\phi\}_m \ell(c^*)_m ds = 0 \tag{7}$$

where ρ is a separation constant, and R_m is the set of numerical control volumes or elements. The assembly of equation (7) is performed over the entire solution domain yielding

$$S_m \left[\int_{R_m} \{\phi\}_m L(c^*)_m d\tau - \rho \int_{\partial R_m \cap \partial R} \{\phi\}_m \ell(c^*)_m ds \right] = 0 \tag{8}$$

where S_m denotes the assembly algorithm, thereby placing the elements of an array into designated locations in an alternative (global) array. Using equations (4), (5), (6), and (8) and integrating the diffusion term by parts give

$$S_m \Bigg[\int_{R_m} \{\phi\}\lfloor\phi\rfloor d\tau \{c\}' + \int_{R_m} \{\phi\}\lfloor\phi\rfloor\{U\} \frac{d}{dx} \lfloor\phi\rfloor d\tau \{c\}$$

$$- \oint_{\partial R_m} \{\phi\} k \frac{d}{dx} \lfloor\phi\rfloor \{c\} \cdot \hat{n} d\tau$$

$$+ \int_{R_m} \frac{d}{dx} \{\phi\} k \frac{d}{dx} \lfloor\phi\rfloor d\tau \{c\}$$

$$- \rho \int \{\phi\} k \frac{d}{dx} \lfloor\phi\rfloor \{c\} \cdot \hat{n} d\tau \Bigg] = \{0\} \qquad (9)$$

If $\rho = -1$ and $a_1 = a_3 = 0$, equation (9) is reduced to

$$S_m \Bigg[\int_{R_m} \{\phi\}\lfloor\phi\rfloor d\tau \{c\}' + \lfloor U\rfloor \int \{\phi\}\{\phi\} \frac{d}{dx} \lfloor\phi\rfloor d\tau \{c\}$$

$$+ k \int \frac{d}{dx} \{\phi\} \frac{d}{dx} \lfloor\phi\rfloor d\tau \{c\} \Bigg] = \{0\} \qquad (10)$$

where $\lfloor\phi\rfloor\{U\} \equiv \lfloor U\rfloor\{\phi\}$.

The problem is now reduced to one of evaluating the integrals appearing in equation (10). This can be easily determined through the use of a natural coordinate system, as shown in Fig. 1. This coordinate system, known as area coordinates in structural mechanics [7], is discussed in more detail by Baker and Soliman [8]. The integral of the arbitrary products of these coordinate functions over a one-dimensional finite element space is analytically established in terms of exponents as

$$\int_{R_m} \xi_1^{n_1} \xi_2^{n_2} dx = L_m \frac{n_1! n_2!}{(1+n_1+n_2)!} \qquad (11)$$

where L_m is the length (span) of the line element. For example,

$$\int \{\phi\}\lfloor\phi\rfloor dx = \int \begin{Bmatrix} \xi_1 \\ \xi_2 \end{Bmatrix} \lfloor \xi_1, \xi_2 \rfloor dx \tag{12}$$

$$= \int \begin{bmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_2\xi_1 & \xi_2^2 \end{bmatrix} dx \tag{13}$$

Thus, from equation (11),

$$\int \{\phi\}\lfloor\phi\rfloor dx = L_m \begin{bmatrix} \frac{2!0!}{(1+2+0)!} & \frac{1!1!}{(1+1+1)!} \\ \frac{1!1!}{(1+1+1)!} & \frac{0!2!}{(1+2+0)!} \end{bmatrix} \tag{14}$$

$$= \frac{L_m}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{15}$$

Likewise, for $\int \{\phi\}\{\phi\} \frac{d}{dx} \lfloor\phi\rfloor dx$, and invoking the chain rule,

$$\frac{d\phi}{dx} = \frac{\partial\phi}{\partial\xi_1} \frac{\partial\xi_1}{\partial x} + \frac{\partial\phi}{\partial\xi_2} \frac{\partial\xi_2}{\partial x} \tag{16}$$

where $\xi_2 = \frac{x-x_1}{L_m}$ and $\xi_1 = \frac{x_2-x}{L_m}$, the advection integral becomes

$$\int \begin{Bmatrix} \xi_1 \\ \xi_2 \end{Bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \end{Bmatrix} \frac{1}{L_m} \lfloor -1, 1 \rfloor dx = \frac{1}{L_m} \int \begin{Bmatrix} \xi_1 \\ \xi_2 \end{Bmatrix} \begin{bmatrix} -\xi_1 & \xi_1 \\ -\xi_2 & \xi_2 \end{bmatrix} dx \tag{17}$$

Equation (17) can be rewritten in hypermatrix form [8] as

$$\frac{1}{L_m} \int \begin{bmatrix} \begin{Bmatrix} -\xi_1^2 \\ -\xi_1\xi_2 \end{Bmatrix} & \begin{Bmatrix} \xi_1^2 \\ \xi_1\xi_2 \end{Bmatrix} \\ \begin{Bmatrix} -\xi_1\xi_2 \\ -\xi_1^2 \end{Bmatrix} & \begin{Bmatrix} \xi_1\xi_2 \\ \xi_2^2 \end{Bmatrix} \end{bmatrix} dx \tag{18}$$

Equation (11) is used to transform equation (18) to

$$\int \{\phi\}\{\phi\} \frac{d}{dx} \lfloor\phi\rfloor dx = \frac{1}{6L_m} \begin{bmatrix} \begin{Bmatrix} -2 \\ -1 \end{Bmatrix} \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} \\ \begin{Bmatrix} -1 \\ -2 \end{Bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \end{bmatrix} \tag{19}$$

The diffusion integral is simply expressed as

$$k\int \frac{d}{dx}\{\phi\}\frac{d}{dx}\{\phi\}dx = k\int \frac{1}{L_m}\begin{Bmatrix}-1\\1\end{Bmatrix}\frac{1}{L_m}\lfloor -1,1\rfloor dx \tag{20}$$

which becomes

$$k\int \frac{d}{dx}\{\phi\}\frac{d}{dx}\lfloor\phi\rfloor dx = \frac{1}{L_m^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \tag{21}$$

Equation (9) is thus transformed to

$$S_m\left(\frac{L_m}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\{c_m\}' + \frac{\lfloor U\rfloor}{6}\begin{bmatrix}\{\begin{smallmatrix}-2\\-1\end{smallmatrix}\}\{\begin{smallmatrix}2\\1\end{smallmatrix}\}\\ \{\begin{smallmatrix}-1\\-2\end{smallmatrix}\}\{\begin{smallmatrix}1\\2\end{smallmatrix}\}\end{bmatrix}\{c_m\} + \frac{k}{L_m}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\{c_m\}\right) = 0 \tag{22}$$

The finite difference recursion relation is established for equation (22) (Fig. 2) by completing the assembly operation.

On the interval from i-1 to i (Δx_-),

$$\frac{L_m}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}c'_{i-1}\\ c'_i\end{Bmatrix} + \frac{[U_{i-1}, U_i]}{6}\begin{bmatrix}\{\begin{smallmatrix}-2\\-1\end{smallmatrix}\}\{\begin{smallmatrix}2\\1\end{smallmatrix}\}\\ \{\begin{smallmatrix}-1 & 1\\-2 & 2\end{smallmatrix}\}\end{bmatrix}\begin{Bmatrix}c_{i-1}\\ c_i\end{Bmatrix}$$

$$+ \frac{k}{L_m}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}c_{i-1}\\ c_i\end{Bmatrix} = \{0\} \tag{23}$$

If the same procedure is applied on the interval i to i+1 (Δx_+), equation (22) becomes

$$\frac{1}{6}\begin{bmatrix}2\Delta x_- & \Delta x_- & 0\\ \Delta x_- & 2\Delta x_- + 2\Delta x_+ & \Delta x_+\\ 0 & \Delta x_+ & 2\Delta x_+\end{bmatrix}\begin{Bmatrix}c'_{i-1}\\ c'_i\\ c'_{i+1}\end{Bmatrix}$$

$$+ \frac{1}{6}\begin{bmatrix}-2U_{i-1}-U_i & 2U_{i-1}+U_i & 0\\ -U_{i-1}-2U_i & U_{i-1}-U_{i+1} & 2U_i+U_{i+1}\\ 0 & -U_i-2U_{i+1} & U_i+2U_{i+1}\end{bmatrix}\begin{Bmatrix}c_{i-1}\\ c_i\\ c_{i+1}\end{Bmatrix}$$

$$+ k\begin{bmatrix}\frac{1}{\Delta x_-} & \frac{-1}{\Delta x_-} & 0\\ \frac{1}{\Delta x_-} & \frac{1}{\Delta x_-}+\frac{1}{\Delta x_+} & \frac{-1}{\Delta x_+}\\ 0 & \frac{-1}{\Delta x_+} & \frac{1}{\Delta x_+}\end{bmatrix}\begin{Bmatrix}c_{i-1}\\ c_i\\ c_{i+1}\end{Bmatrix} = 0 \tag{24}$$

The equivalent difference-differential equation for node i is

$$\frac{1}{6}\left[c'_{i-1}\Delta x_- + 2c'_i(\Delta x_- + \Delta x_+) + c'_{i+1}\Delta x_+\right]$$

$$+ \frac{1}{6}\left[c_{i-1}(-U_{i-1} - 2U_i) + c_i(U_{i-1} - U_{i+1}) + c_{i+1}(2U_i + U_{i+1})\right]$$

$$+ k\left[-\frac{c_{i-1}}{\Delta x_-} + c_i\left(\frac{1}{\Delta x_-} + \frac{1}{\Delta x_+}\right) - \frac{c_{i+1}}{\Delta x_+}\right] = 0 \qquad (25)$$

Equation (25) holds for variable element length and nonuniform velocity and is tridiagonal, i.e., involves only i-1, i, and i+1 grid points. If $\Delta x_- = \Delta x_+$ and the velocity is constant, equation (25) becomes

$$\frac{1}{6}\left[c'_{i-1} + 4c'_i + c'_{i+1}\right] + \frac{U}{2\Delta x}\left[c_{i+1} - c_{i-1}\right]$$

$$- \frac{k}{\Delta x^2}\left[c_{i+1} - 2c_i + c_{i-1}\right] = 0 \qquad (26)$$

By expressing the time derivative appearing in equation (26) by an implicit one-step difference algorithm, equation (26) can be rewritten as

$$\frac{1}{6}\left[(c^{n+1}_{i-1} - c^n_{i-1}) + 4(c^{n+1}_i - c^n_i) + (c^{n+1}_{i+1} - c^n_{i+1})\right]$$

$$+ \frac{U}{2\Delta x}\left[\mu(c_{i+1} - c_{i-1})^{n+1} + (1-\mu)(c_{i+1} - c_{i-1})^n\right] \qquad (27)$$

$$- \frac{k}{\Delta x^2}\left[\mu(c_{i+1} - 2c_i + c_{i-i})^{n+1} + (1-\mu)(c_{i+1} - 2c_i + c_{i-1})^n\right] = 0$$

where superscript n denotes present value at time level n, and μ is the time integration parameter, $0 < \mu < 1$. For $\mu = 1$, equation (27) is fully implicit; for $\mu = 0$, equation (27) is explicit.

Best results are obtained for $\mu = 1/2$ (Crank-Nicolson). Since the advection terms are crucial to the computational stability of equation (27), an examination of the amplification factor associated with equation (27) can be easily made. The diffusion term is of secondary importance since it tends to stabilize (and damp) the distribution with time. By the procedure outlined by Roache [9], the relation $g^n \exp(i\lambda j\Delta x)$ is substituted into equation (27). Here g is the amplification factor, $j = \sqrt{-1}$, and λ is the wavenumber ($\lambda = 2\pi/L$) of the Fourier component at time level n. The amplification factor associated with equation (27) is

$$g = \frac{1 + \frac{\cos(\lambda\Delta x)}{2}[2 + (1-\mu)\alpha] - (1-\mu)\alpha - \frac{3\sigma(1-\mu)j}{2}\sin(\lambda\Delta x)}{1 + \frac{\cos(\lambda\Delta x)}{2}[1 - \mu\alpha] + \mu\alpha + \frac{3\sigma\mu j}{2}\sin(\lambda\Delta x)} \tag{28}$$

where $\sigma = U\Delta t/\Delta x$ and $\alpha = k\Delta t/2\Delta x^2$.

Setting $\alpha = 0$ and $\mu = 1/2$ reduces equation (28) to

$$g = \frac{1 + \frac{\cos(\lambda\Delta x)}{2} - \frac{3\sigma j}{4}\sin(\lambda\Delta x)}{1 + \frac{\cos(\lambda\Delta x)}{2} + \frac{3\sigma j}{4}\sin(\lambda\Delta x)} \tag{29}$$

which represents the amplification factor for the one-dimensional advection equation. Since the numerator and denominator are complex conjugates of each other in equation (29), $g = 1$ for all Δx, Δt, and λ. Hence, there is neither damping nor amplification of waves. If $\mu < 1/2$, the solution becomes amplified; if $\mu > 1/2$, heavy damping occurs. An analysis of the phase speed of equation (29) along

with speeds of several standard finite difference schemes is discussed in detail by Long and Hicks [2]. By performing a Taylor series expansion about x_i, equation (27) (with k = 0) can be transformed to the relation

$$\frac{\partial c}{\partial t} + U \frac{\partial c}{\partial x} - \frac{U\Delta x^4}{180} \frac{\partial^5 c}{\partial x^5} + 0\ (\Delta t^2, \Delta x^5) \tag{30}$$

which shows the scheme to be fourth order accurate for uniform element length and constant velocity.

The technique can be extended to two- or three-dimensional space by writing a series of one-dimensional equations. For example, the three-dimensional advection-diffusion equation is split into three one-dimensional equations such that

$$\frac{\partial c^{\dagger}}{\partial t} + U \frac{\partial c^n}{\partial x} - \frac{\partial}{\partial x} \left(k_x \frac{\partial c^n}{\partial x}\right) = 0 \tag{31}$$

$$\frac{\partial c^{\dagger\dagger}}{\partial t} + V \frac{\partial c^{\dagger}}{\partial y} - \frac{\partial}{\partial y} \left(k_y \frac{\partial c^{\dagger}}{\partial y}\right) = 0 \tag{32}$$

$$\frac{\partial c^{n+1}}{\partial t} + W \frac{\partial c^{\dagger\dagger}}{\partial z} - \frac{\partial}{\partial z} \left(k_z \frac{\partial c^{\dagger\dagger}}{\partial z}\right) = 0 \tag{33}$$

The time-splitting procedure is discussed by Yanenko [10]. Equations (25) and (31-33) have been used by Pepper and Kern [6] and Long and Pepper [4] to model the three-dimensional transport of atmospheric pollutant.

APPLICATION AND RESULTS

Equation (27) is used to solve the advection (k = 0) of a "cosine hill" passive scalar in one dimension. Fig. 3a shows

the initial distribution at t = 0. Fig. 3b shows the analytical solution comparison with the solution obtained with equation (27). The Courant number is equal to 0.4. For small Courant numbers, the comparison between the two solutions is nearly perfect. The small wake created by the finite element solution is due to the inability of the finite element method to resolve $2\Delta x$ waves. Raymond and Garder [1] and Pepper and Kern [6] discuss ways of eliminating dispersive wakes. As the Courant number increases, more dispersion occurs.

The inclusion of diffusion is shown in Fig. 3c and 3d for $\alpha = 0.01$ and $\alpha = 0.0001$ respectively. The lateral spreading and peak damping of the distribution occur as expected due to the inclusion of diffusion (as α increases, longer waves are damped).

The advection of a two-dimensional cosine hill is shown in Fig. 4. The Courant numbers in the x and y directions are equal to 0.10. The initial distribution is shown in Fig. 4a with a subsequent distribution shown at t = 150 in Fig. 4b. In this test, equation (27) was solved twice: first in the x direction and then in the y direction (analogous to an ADI technique). The mesh size is 33 × 33. Core requirements were minimal (since only a tridiagonal solution is required at one time), and computational running time was about 10 sec on an IBM 360/195. The distribution is not significantly altered at t = 150; it should ideally remain the same as at t = 0. Solution of equation (25)

for variable velocity and nonuniform grid network in two- and three-dimensional space is discussed by Pepper and Kern [6].

The use of higher degree interpolation polynomials, such as quadratic and cubic elements, does not lend itself directly to establishment of a simple ~~finite difference~~ tri-diagonal recursion relation. However, These functions do yield higher order accurate algorithms (Baker and Soliman [8]), which are readily programmed with the parent finite element algorithm, equation (10).

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CAPTIONS FOR FIGURES

Fig. 1. Natural coordinate system.

Fig. 2. Assemblage over two adjacent elements.

Fig. 3. Advection of a concentration packet in one dimension with $\sigma = 0.4$. (Dashed line is analytical solution.)

Fig. 4. Two-dimensional advection of a cosine hill passive scalar.

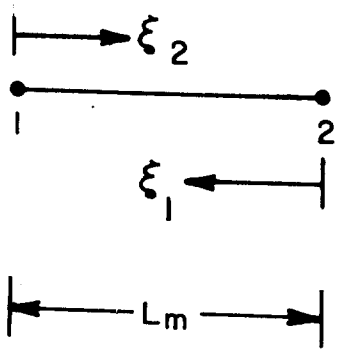

Fig. 1. Natural coordinate system.

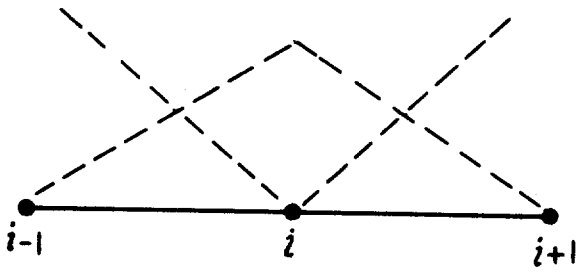

Fig. 2. Assemblage over two adjacent elements.

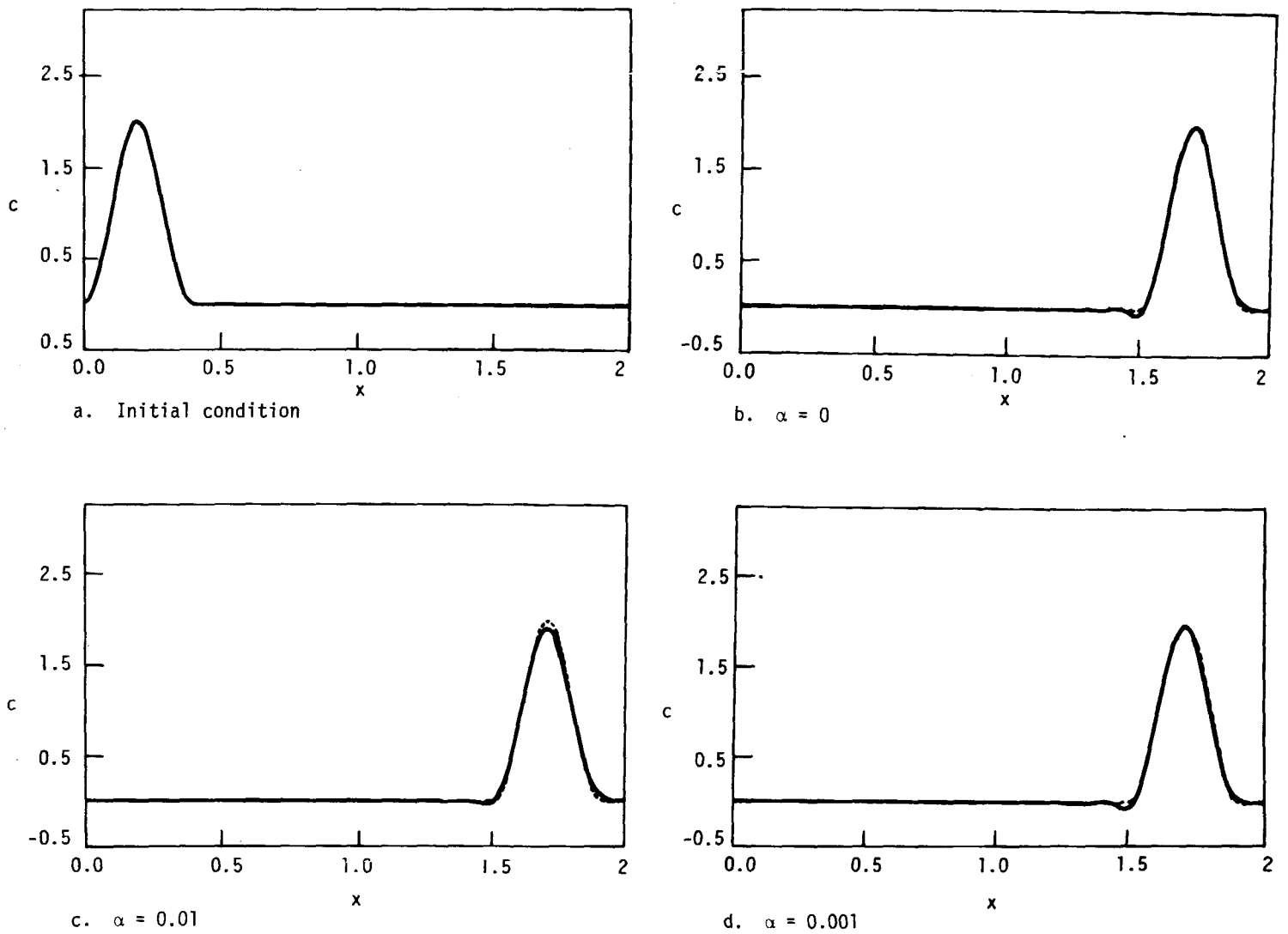

Fig. 3. Advection of a concentration packet in one dimension with $\sigma = 0.4$. (Dashed line is analytical solution.)

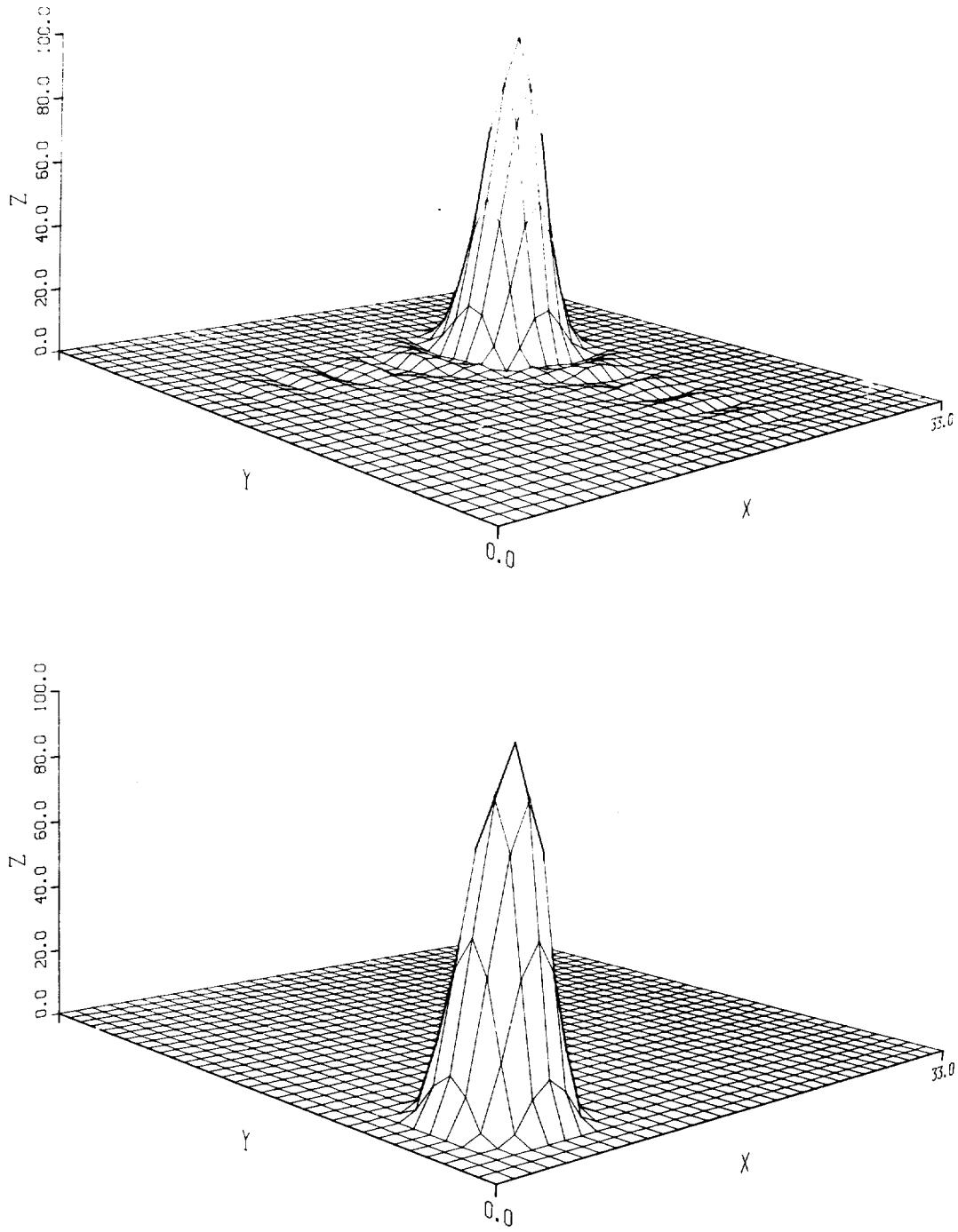

Fig. 4. Two-dimensional advection of a cosine hill passive scalar.